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LETTER TO THE EDITOR

Absorption time in certain urn models

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Abstract. We show that the average absorption time τ in a certain M-urn model recently introduced by Ritort in the context of glassy dynamics has a lower bound $2^{N-1}+N-3$, where N is the number of balls. A Monte Carlo scheme, which accurately estimates the absorption time even when $\tau \sim 10^{12}$, indicates that for large N we have $\tau \sim 2^N$.

The urn models have played a very important role in formulating such fundamental concepts of statistical mechanics such as equilibrium, an approach to equilibrium or fluctuations out of equilibrium [1]. One of the most intensively studied urn models was the Ehrenfest model ('dog–flea' model). In this model distinguishable balls are placed in two urns. In each time step a randomly chosen ball is taken from one urn and put into the other one. In this model, which is actually equivalent to a certain random walk, a lot of quantities can be calculated exactly [2].

Recently, Ritort has introduced a multistate Potts model which describes slow dynamics encountered in glasses [3]. At the ground state the dynamics of this model is equivalent to a certain M-urn model which in turn can be mapped into a biased random walk [4]. This M-urn model is defined as follows. Let us consider N balls initially distributed among M boxes. At each time step a randomly chosen ball is moved to another non-empty box chosen randomly.

It is clear that once a box becomes empty it remains such forever and thus the number of non-empty boxes decreases until all balls are placed in one box. It is also easy to realize that the number of non-empty boxes decreases with a slower and slower rate because emptying a box with a few balls and placing them into boxes of larger occupancy is a highly unlikely event.

It is this slow dynamics which bears some resemblance to the dynamics of glasses. This model shows that entropic barriers alone might be responsible for the slow dynamics but in real glasses one expects that energetic barriers are also present and relevant.

One of the important characteristics of this model is the average absorption time τ which is needed to put all balls in one box; this quantity corresponds to the time needed to bring a glass to the ground state. Ritort's Monte Carlo results, based on simulations with $N \leqslant 20$, suggest that for large N one has $\tau \sim \mathrm{e}^{0.67N}$.

In the present letter we obtain a lower bound for τ which is in fairly good agreement with Ritort's Monte Carlo results. Our lower bound is actually based on the exact solution for the absorption time for the 2-urn model. To our knowledge this is a new result. Moreover, the exact solution for the 2-urn model (given as a recursive sequence) enabled us to construct

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a Monte Carlo scheme which precisely measures τ for systems even twice as large as those considered by Ritort.

First let us consider the case $M = 2$ and the initial configuration with one ball in the first urn and $N - 1$ balls in the second urn. In such a case the average time $\tau_2(1, N-1)$ needed to place all balls in one box is

$$\tau_2(1, N-1) = 2^{N-1} - 1. \tag{1}$$

This result will be proven below.

However, when balls are initially distributed among $M > 2$ boxes, the absorption time $\tau_M(N)$ has to be strictly greater than equation (1). Indeed, since such a process has to pass through the configuration at least once with all but one ball in one box we have

$$\tau_M(N) = \tau'_M(N) + \tau_2(1, N-1) \tag{2}$$

where $\tau'_M(N) > 0$ is the average time needed to reach the configuration with all but one ball in one box. For example, if initially there is one ball in each of N boxes, then the lower bound for $\tau'_N(N)$ is $N - 2$ and in such a case we obtain

$$\tau_N(N) > 2^{N-1} + N - 3. \tag{3}$$

Now we proceed to determine $\tau_2(k, N-k)$, i.e. the absorption time for the initial configuration of k and $N - k$ balls in each urn. First, let us consider the case $N = 3$. In this case $\tau_2(1, 2)$ has to satisfy the equation

$$\tau_2(1,2) = \tfrac{1}{3} \cdot 1 + \tfrac{2}{3}(\tau_2(1,2) + 1). \tag{4}$$

The right-hand side of (4) means that with the probability $\frac{1}{3}$ the process will terminate (i.e. a ball from the box occupied by only one ball will be moved to the other box) and with the probability $\frac{2}{3}$ the system will remain in the same configuration (with the occupancy of boxes interchanged). The solution of (4) is $\tau_2(1, 2) = 3$, in agreement with (1).

For $N = 4$ the following set of equations is the analogue of (4)

$$\begin{aligned} \tau_2(1,3) &= \tfrac{1}{4} \cdot 1 + \tfrac{3}{4}(\tau_2(2,2) + 1) \\ \tau_2(2,2) &= \tau_2(1,3) + 1 \end{aligned} \tag{5}$$

with the solution $\tau_2(1, 3) = 7$ and $\tau_2(2, 2) = 8$.

In the case of arbitrary even N we obtain the following set of $N/2$ equations:

$$\begin{aligned} \tau_2(1, N-1) &= \frac{1}{N} + \frac{N-1}{N}(\tau_2(2, N-2) + 1) \\ \tau_2(2, N-2) &= \frac{2}{N}(\tau_2(1, N-1) + 1) + \frac{N-2}{N}(\tau_2(3, N-3) + 1) \\ &\cdots \\ \tau_2(k, N-k) &= \frac{k}{N}(\tau_2(k-1, N-k+1) + 1) + \frac{N-k}{N}(\tau_2(k+1, N-k-1) + 1) \\ &\cdots \\ \tau_2\left(\frac{N}{2}, \frac{N}{2}\right) &= \tau_2\left(\frac{N}{2} - 1, \frac{N}{2} + 1\right) + 1. \end{aligned} \tag{6}$$

For odd N the set of equations is the same except that the last equation has the form

$$\begin{aligned} \tau_2\left(\frac{N-1}{2}, \frac{N+1}{2}\right) = \frac{N-1}{2N}&\left(\tau_2\left(\frac{N-3}{2}, \frac{N+3}{2}\right) + 1\right) \\ &+ \frac{N+1}{2N}\left(\tau_2\left(\frac{N-1}{2}, \frac{N+1}{2}\right) + 1\right). \end{aligned} \tag{7}$$

We did not succeed in writing a solution of (6) in a compact form for arbitrary $\tau_2(k, N-k)$. However, for $\tau_2(1, N-1)$ the solution can be found easily. One can check that multiplying the first equation by N, the second one by $N(N-1)/2 = \binom{N}{2}$, ..., the last but one by $\binom{N}{N-1}$, and the last one by $\frac{1}{2}\binom{N}{N/2}$, and summing all equations we obtain that all $\tau_2(k, N-k)$ with $k > 1$ cancel and the remaining terms give

$$\tau_2(1, N-1) = \binom{N}{1} + \binom{N}{2} + \cdots + \binom{N}{(N/2)-1} + \frac{1}{2}\binom{N}{N/2} = 2^{N-1} - 1. \tag{8}$$

Once we have found $\tau_2(1, N-1)$ we can easily write the recursive solution of (6):

$$\begin{aligned}
\tau_2(1, N-1) &= 2^{N-1} - 1 \\
\tau_2(2, N-2) &= \frac{N(\tau_2(1, N-1) + 1) - 2N}{N-1} \\
\tau_2(k, N-k) &= \frac{N\tau_2(k-1, N-k+1) - (k-1)\tau_2(k-2, N-k+2) - N}{N-k+1}
\end{aligned} \tag{9}$$

where $k = 3, 4, \ldots, N/2$.

To confront the bound (3) with the behaviour of $\tau_N(N)$ we performed Monte Carlo simulations. However, using the exact solution (9) we were able to make our algorithm very efficient. The basic idea of this hybrid algorithm is as follows. Initially, each N boxes is occupied by one ball. Then we simulate the dynamics of the system until there are two non-empty boxes. Provided that their occupancy at the moment of reaching such a situation is k and $N-k$, we add to the time of this particular run the average time needed to terminate the process, i.e. $\tau_2(k, N-k)$ (these numbers can be generated at the beginning of the program and stored in a memory). For $N \leqslant 20$ we checked our algorithm against the ordinary algorithm which simulates the dynamics until all balls are placed in one box and we obtained very good agreement.

Since the absorption time rapidly increases with N, simulating larger systems with the ordinary algorithm is rather difficult. In contrast, using the hybrid algorithm we were able to simulate systems up to $N = 40$ for which the absorption time is of the order of 10^{12}(!). Our simulations have been performed on a personal computer and simulating systems with larger N is obviously feasible.

The reason why this algorithm is so efficient is that, as will be shown below, for large N the time needed to decrease the number of non-empty boxes to two becomes negligibly small compared to the time needed to terminate the process and our program runs only in the first interval.

Our results are shown in figure 1. For $N < 20$ averages are made over 10^6 independent runs. For larger N we usually average over 10^4 runs but in this case even less extensive statistics gives accurate results. This is because for large N dominant contributions to $\tau_N(N)$ come from $\tau_2(k, N-k)$ and, as we already mentioned, we calculate this number exactly.

An almost linear increase of the logarithm of $\tau_N(N)$ as a function of N confirms its exponential divergence. Least-squares fitting to the results with $N = 20, \ldots, 40$ gives $\tau_N(N) \sim 0.54\,\mathrm{e}^{0.693N}$, which as for the exponential part is in fairly good agreement with Ritort's result [3]. However, the fact that $\mathrm{e}^{0.693} \sim 1.9997\ldots$ strongly suggests that for large N the exponential part is the same as in our lower bound (3). Using this assumption we analysed the quantity $r = \tau_N(N)/2^N$ as a function of $1/N$ and we found that for large N we have $r = (1/2) + (\alpha/N)$ where $\alpha \sim 0.5$. Thus, the asymptotic behaviour of $\tau_N(N)$ is $\tau_N(N) = (1 + (\alpha/N))2^{N-1}$. The main conclusion which one can draw from our analysis is

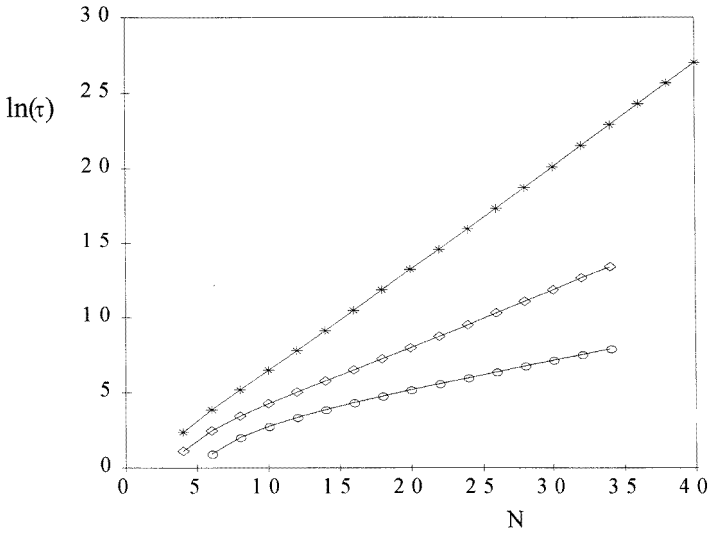

Figure 1. Logarithm of the absorption time $\tau_N(N)$ ($*$) as a function of N. We also present the logarithm of the average time needed to place N balls in two boxes $\tau^{(2)}$ ($\diamond$) and in four boxes $\tau^{(4)}$ ($\bigcirc$) plotted as a function of N.

that the time scale of the absorption time in Ritort's model is the same as that in the 2-urn model.

To show that putting all N balls in one box is indeed the longest process we also measured the averaged time needed to place all balls in two non-empty boxes ($\tau^{(2)}$) and in four non-empty boxes ($\tau^{(4)}$). Logarithms of these quantities as functions of N are also shown in figure 1. From figure 1 it is clear that $\tau^{(2)}$ and $\tau^{(4)}$ also diverge exponentially with N but much slower than $\tau_N(N)$. Our estimations are $\tau^{(2)} \sim 1.18\,\mathrm{e}^{0.390N}$ and $\tau^{(4)} \sim 3.55\,\mathrm{e}^{0.196N}$.

The fact that $\tau^{(2)}$ becomes negligibly small comparing to $\tau_N(N)$ explains why the hybrid algorithm is much faster than standard simulations (for $N = 40$ the efficiency increases about 65 000 times and standard simulations are simply impossible).

Although static properties of Ritort's model are very simple and solvable, its dynamics which shows quite interesting behaviour most likely cannot be studied exactly [3, 4]. Our analysis shows that an important property of Ritort's model, namely exponentially diverging absorption time, appears actually in a much simpler model. It is thus interesting to examine if one can construct a simpler analogue of Ritort's model which would be solvable also with respect to the dynamics.

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